Aerodynamic exam 080716

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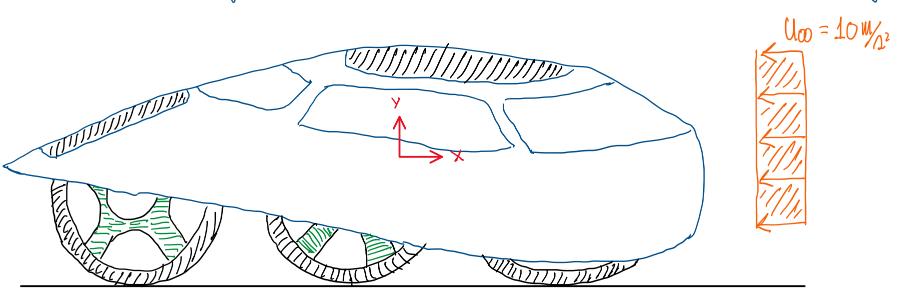
A solar energy powered vehicle has a streamlined body and three wheels partially hiden by the vehicle body. The downforce coefficient is $C_L = 1$ and the vertical loads are equally distributed over the wheels. The vehicle mass is $m = 480\ Kg$, the frontal area is $A = 1\ m^2$, the lateral friction coefficient is $K_a = 1.0$, the gravity acceleration is $g = 9.81\ m/s^2$. The velocity profile of the car respective to the longitudinal symmetry plane is given by the following equation:

$$\frac{u}{U_\infty} = (\eta-1)^3\cdot(\eta+1) - \beta(x)\cdot\eta\cdot(\eta-1)^3 + 1 \quad ; \quad \eta = \frac{z}{\delta(x)}$$

where z is the vertical coordinate normal the vehicle body, $\delta(x)$ is boundary layer thickness and $\beta(x)$ is a decreasing function of x that accounts the evolution of the velocity profile.



1 - Calculate the minimum curvature radius when the vehicle velocity is $U_\infty = 10\ m/s^2$: There are two important aspects when analyzing the vehicle capacity to go around a corner, one is the tire performance, the other is the aerodynamics. Actually, tires depend of their contacts with the surface. Since there is an aerodynamic in this vehicle, the friction coefficient gives place to the aerodynamic rigidity, K_a, which is given by $K_a = -q_\infty \cdot A \cdot \partial C_z/\partial h$, thus it is a function of the vehicle ride height, frontal area and the dynamic pressure. Hence, this aerodynamic stiffness depends on velocity, because the dynamic pressure is velocity dependent. Although $\partial C_z/\partial h$ also is velocity dependent, this proportion also is connected with other parameters. Therefore the calculations are:

$$q_\infty = \frac{1}{2}\rho\cdot U_\infty^2 = \frac{1}{2}\,1.2\cdot 10^2\left[\frac{Kg}{m^3}\cdot\left(\frac{m}{s}\right)^2\right] = 60\ Pa$$

$$\mu\cdot(mg - L) = m\cdot\frac{u^2}{R} \rightarrow R = \frac{m\cdot u^2}{\mu(m\cdot g - L)}$$

$$L = q_\infty\cdot C_L\cdot A = -60\cdot 1\cdot 1\ [Pa\cdot m^2] = -60\ N \quad \text{(DOWNFORCE)}$$

$$R = \frac{480\cdot 10^2}{1\cdot(480\cdot 10 + 60)} = \frac{48000}{4860} = 9.877\ m$$

In these calculations it was possible to notice two equations to calculates the lateral force in a vehicle, $\mu(m\cdot g - L)$ and $m\cdot u^2\cdot R^{-1}$. This last one is the component defined by the vehicle dynamics and geometry. So this is limit that the car is not able to overcome. Hence, the first component should equal or higher the geometric limit. Finally, it is observed that there is a maximum curvature radius that the car can go. Another conclusion is that this car has a streamlined bodywork, this produces a very small lift, but it still has some impact in the vehicle cornering capability. Actually it produced some amount of downforce.

2 - Calculate $\beta(x)$ at separation point: In this case it is important to understand how the flow behaves over a streamlined body. Considering the following equation, valid for wing profiles under ground effect and low attack angle α:

$$C_L = 2\pi\cdot\left[1 + \left(\frac{c}{4h}\right)^2\right]\alpha \quad ; \quad [\alpha] = RAD$$

The point is, the position that separation occurs is given by the component the velocity gradient is vanished:

$$\frac{\partial u}{\partial y}(x_s, y=0) = 0$$

Therefore the following procedure is performed:

$$u(\eta) = U_\infty\left[(\eta-1)^3\cdot(\eta+1) - \beta(x)\cdot\eta\cdot(\eta-1)^3 + 1\right] \quad ; \quad \eta = \frac{z}{\delta(x)}$$

$$u(\eta) = U_\infty\left[\left(\frac{z}{\delta(x)}-1\right)^3\left(\frac{z}{\delta(x)}+1\right) - \beta(x)\cdot\frac{z}{\delta(x)}\cdot\left(\frac{z}{\delta(x)}-1\right)^3 + 1\right]$$

$$u(\eta) = U_\infty\left[\left(\frac{z^2}{\delta(x)^2} - 2\frac{z}{\delta(x)} + 1\right)\cdot\left(\frac{z}{\delta(x)}-1\right)\cdot\left(\frac{z}{\delta(x)}+1\right) - \beta(x)\frac{z}{\delta(x)}\left(\frac{z^3}{\delta^3(x)} - 3\frac{z^2}{\delta^2(x)} + 3\frac{z}{\delta(x)} - 1\right)\right]$$

$$u(\eta) = U_\infty\left[\left(\frac{z^3}{\delta(x)^3} - \frac{z^2}{\delta^2(x)} - 2\frac{z^2}{\delta(x)^2} + 2\frac{z}{\delta(x)} + \frac{z}{\delta(x)} - 1\right)\cdot\left(\frac{z}{\delta(x)}+1\right)\right.$$
$$\left. - \beta(x)\cdot\frac{z}{\delta(x)}\cdot\left(\frac{z^3}{\delta(x)^3} - \frac{z^2}{\delta(x)^2} - 2\cdot\frac{z^2}{\delta(x)^2} + 2\frac{z}{\delta(x)} - 1 + \frac{z}{\delta(x)}\right)\right]$$

$$u(\eta) = U_\infty\left[\left(\frac{z^3}{\delta(x)^3} - 3\cdot\frac{z^2}{\delta(x)^2} + 3\cdot\frac{z}{\delta(x)} - 1\right)\cdot\left(\frac{z}{\delta(x)}+1\right) - \beta(x)\frac{z}{\delta(x)}\left(\frac{z^3}{\delta(x)^3} - 3\frac{z^2}{\delta(x)^2} + 3\frac{z}{\delta(x)} - 1\right)\right]$$

$$u(\eta) = U_\infty\left[\left(\frac{z^4}{\delta(x)^4} + \frac{z^3}{\delta(x)^3} - 3\frac{z^3}{\delta(x)^3} - 3\frac{z^2}{\delta(x)^2} + 3\frac{z^2}{\delta(x)^2} + 3\frac{z}{\delta(x)} - \frac{z}{\delta(x)} - 1\right)\right.$$
$$\left. - \beta(x)\cdot\left(\frac{z^4}{\delta(x)^4} - 3\cdot\frac{z^3}{\delta(x)^3} + 3\frac{z^2}{\delta(x)^2} - \frac{z}{\delta(x)}\right)\right]$$

$$u(\eta) = U_\infty\left[\left(\frac{z^4}{\delta(x)^4} - 2\frac{z^3}{\delta(x)^3} + 2\frac{z}{\delta(x)} - 1\right) - \beta(x)\left(\frac{z^4}{\delta(x)^4} - 3\frac{z^3}{\delta(x)^3} + 3\frac{z^2}{\delta(x)} - \frac{z}{\delta(x)}\right)\right]$$

$$\frac{\partial u}{\partial z} = U_\infty\left[\left(4\frac{z^3}{\delta(x)^4}\xrightarrow{}0 - 2\cdot 3\frac{z^2}{\delta(x)^3}\xrightarrow{}0 + 2\frac{z}{\delta(x)}^{\,0} + 2\frac{1}{\delta(x)}\right)\right.$$
$$\left. - \beta(x)\left(4\cdot\frac{z^3}{\delta(x)^4}\xrightarrow{}0 - 3\cdot 3\cdot\frac{z^2}{\delta(x)^3}\xrightarrow{}0 + 3\cdot 2\cdot\frac{z}{\delta(x)^2}\xrightarrow{}0 - \frac{1}{\delta(x)}\right)\right]$$

$$\frac{\partial u}{\partial z} = U_\infty\left[2\frac{1}{\delta(x)} - \beta(x)\cdot\left(-\frac{1}{\delta(x)}\right)\right] = 0$$

$$2\frac{1}{\delta(x)} = -\beta(x)\cdot\frac{1}{\delta(x)}$$

$$\beta(x) = -2$$

As can be seen, $\beta(x) = -2$, because the definition of separation establishes that it occurs just at the wall. The value of -2 indicates that there is separation, since $\beta(x)$ is a decreasing function that indicates a posible boundary layer separation. Hence, $\beta(x)$ must be negative to occur separation.

3 - Indicate the $\beta(x)$ range which the pressure gradient at boundary layer is adverse: This question proposes the definition of the range of $\beta(x)$, which the pressure gradient is adverse. This is defined by the curvature of the velocity profile $\beta(x)$, its sign is the same as the pressure gradient. Hence, to calculate $\beta(x)$ which the pressure gradient is adverse:

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_w = \frac{1}{\mu}\cdot\left.\frac{dP}{dx}\right|_w \approx \frac{1}{\mu}\cdot\frac{dP}{dx}$$

For separation, the pressure gradient must be adverse, which means that the condition can be written as:

$$\frac{\partial P}{\partial x} > 0$$

Hence, the calculations are:

$$\frac{u}{U_\infty} = (\eta-1)^3\cdot(\eta+1) - \beta(x)\cdot\eta\cdot(\eta-1)^3 + 1 \quad ; \quad \eta = \frac{z}{\delta(x)}$$

$$u = U_\infty\left[\underbrace{\left(\frac{z}{\delta(x)}-1\right)^3\left(\frac{z}{\delta(x)}+1\right)}_{①} - \beta(x)\cdot\underbrace{\left(\frac{z}{\delta(x)}\right)\cdot\left(\frac{z}{\delta(x)}-1\right)^3}_{②} + 1\right]$$

① $$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z}{\delta(x)}-1\right)\left(\frac{z}{\delta(x)}-1\right)\left(\frac{z}{\delta(x)}-1\right) = \left(\frac{z^2}{\delta(x)^2} - \frac{z}{\delta(x)} - \frac{z}{\delta(x)} + 1\right)\left(\frac{z}{\delta(x)}+1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z^2}{\delta(x)^2} - 2\frac{z}{\delta(x)} + 1\right)\left(\frac{z}{\delta(x)}-1\right) = \left(\frac{z^3}{\delta(x)^3} - \frac{z^2}{\delta(x)^2} - 2\frac{z^2}{\delta(x)^2} + 2\frac{z}{\delta(x)} + \frac{z}{\delta(x)} - 1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z^3}{\delta(x)^3} - 3\frac{z^2}{\delta(x)^2} + 3\frac{z}{\delta(x)} - 1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3\cdot\left(\frac{z}{\delta(x)}+1\right) = \frac{z^4}{\delta(x)^4} + \frac{z^3}{\delta(x)^3} - 3\frac{z^3}{\delta(x)^3} - 3\frac{z^2}{\delta(x)^2} + 3\frac{z^2}{\delta(x)^2} + 3\frac{z}{\delta(x)} - \frac{z}{\delta(x)} - 1$$

$$\left(\frac{z}{\delta(x)}-1\right)^3\cdot\left(\frac{z}{\delta(x)}+1\right) = \left(\frac{z^4}{\delta(x)^4} - 2\frac{z^3}{\delta(x)^3} + 2\frac{z}{\delta(x)} - 1\right)$$

② $$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z}{\delta(x)}-1\right)\left(\frac{z}{\delta(x)}-1\right)\left(\frac{z}{\delta(x)}-1\right) = \left(\frac{z^2}{\delta(x)^2} - \frac{z}{\delta(x)} - \frac{z}{\delta(x)} + 1\right)\left(\frac{z}{\delta(x)}+1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z^2}{\delta(x)^2} - 2\frac{z}{\delta(x)} + 1\right)\left(\frac{z}{\delta(x)}-1\right) = \left(\frac{z^3}{\delta(x)^3} - \frac{z^2}{\delta(x)^2} - 2\frac{z^2}{\delta(x)^2} + 2\frac{z}{\delta(x)} + \frac{z}{\delta(x)} - 1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3 = \left(\frac{z^3}{\delta(x)^3} - 3\frac{z^2}{\delta(x)^2} + 3\frac{z}{\delta(x)} - 1\right)$$

$$\left(\frac{z}{\delta(x)}-1\right)^3\cdot\left(\frac{z}{\delta(x)}\right) = \frac{z^4}{\delta(x)^4} - 3\frac{z^3}{\delta(x)^3} + 3\frac{z^2}{\delta(x)^2} + \frac{z}{\delta(x)}$$

$$u = U_\infty\left[\left(\frac{z^4}{\delta(x)^4} - 2\frac{z^3}{\delta(x)^3} + 2\frac{z}{\delta(x)} - 1\right) - \beta(x)\left(\frac{z^4}{\delta(x)^4} - 3\frac{z^3}{\delta(x)^3} + 3\frac{z^2}{\delta(x)^2} + \frac{z}{\delta(x)}\right) + 1\right]$$

$$\frac{\partial u}{\partial z} = U_\infty\left[\left(4\frac{z^3}{\delta(x)^4} - 6\frac{z^2}{\delta(x)^3} + 2\frac{1}{\delta(x)}\right) - \beta(x)\left(4\frac{z^3}{\delta(x)^4} - 9\frac{z^2}{\delta(x)^3} + 6\frac{z}{\delta(x)^2} + \frac{1}{\delta(x)}\right)\right]$$

$$\frac{\partial^2 u}{\partial z^2} = U_\infty\left[\left(12\frac{z^2}{\delta(x)^4} - 12\frac{z}{\delta(x)^3}\right) - \beta(x)\left(12\frac{z^2}{\delta(x)^4} - 18\frac{z}{\delta(x)^3} + 6\frac{1}{\delta(x)^2}\right)\right]$$

$$\frac{\partial^2 u}{\partial z^2} = U_\infty\left(-\beta(x)\cdot 6\frac{1}{\delta(x)^2}\right) = -\beta(x)\cdot\frac{6\cdot U_\infty}{\delta(x)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = -\beta(x)\cdot\frac{6\cdot U_\infty}{\delta(x)^2}$$

Since the conditions for the adverse pressure gradient are $dP/dx > 0$ and $du/dx < 0$:

$$\left.\frac{\partial^2}{\partial y^2}\right|_w = \frac{1}{\mu}\cdot\left.\frac{dP}{dx}\right|_w \approx \frac{1}{\mu}\cdot\frac{dP_e}{dx}$$

$$-\beta(x)\cdot\frac{6\cdot U_\infty}{\delta(x)^2} = \frac{1}{\mu}\cdot\frac{dP}{dx} \rightarrow \frac{dP}{dx} = -\beta(x)\cdot\frac{6\cdot\mu\cdot U_\infty}{\delta(x)^2} \quad ; \quad \beta(x) < 0$$

$\beta(x)$ should be negative, because U_∞ and $\delta(x)^2$ are positive quantities, thus the previous result of $\beta(x) = -2$ was correct.

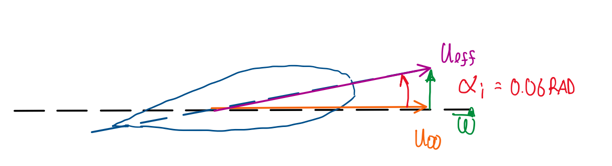
4 - Calculate the attack angle for a chord $c = 3.2\ m$ and ride height $h = 0.4\ m$: Considering that the wing is at ground effect and under low attack angle, the equation for the lift coefficient is given by:

$$C_L = 2\pi\left[1 + \left(\frac{c}{4h}\right)^2\right]\alpha = 2\pi\left[1 + \left(\frac{3.2}{4\cdot 0.4}\right)^2\right]\alpha = 10\pi\cdot\alpha$$

$$10\pi\,\alpha = 1.0$$

$$\alpha = \frac{1}{10\pi} \cong 0.032\ RAD$$

Now it is possible to calculate the vehicle drag since $\alpha_i = 0.06\ RAD$. Assuming that the streamlined body of this car has a similar profile to an elliptical wing.



$$\alpha_i = \frac{C_L}{\pi AR} \rightarrow AR = \frac{C_L}{\pi\cdot\alpha_i} \rightarrow C_{D,i} = \frac{C_L^2}{\pi AR} = \frac{C_L^2}{\pi\cdot\frac{C_L}{\pi\cdot\alpha_i}} = C_L\cdot\alpha_i$$

$$C_{D,i} = C_L\cdot\alpha_i = 1\cdot 0.06 = 0.06$$

$$D_i = q_\infty\cdot A\cdot C_{D_i} = \frac{1}{2}\rho\cdot U_\infty^2\cdot c\cdot C_{D,i} = \frac{1}{2}\cdot 1.2\cdot 10^2\cdot 3.2\cdot 0.06 = 11.52\ \frac{N}{m}$$

Now, the downwash wind component can be defined according to the induced angle of attack illustrated at figure:

$$\alpha_i = \arctan\left(\frac{-w}{U_\infty}\right) \approx \frac{-w}{U_\infty} \rightarrow w = -U_\infty\cdot\alpha_i = -10\cdot 0.06$$

$$\vec{w} = -0.6\ m/s$$